# Completely Antisymmetric Torsion in Conformal Gravitation with Electrodynamics for Dirac Spinors

Luca Fabbri

DIPTEM Sez. Metodi e Modelli Matematici, Università di Genova,

Piazzale Kennedy Pad. D, 16129 Genova, ITALY and

INFN & Dipartimento di Fisica, Università di Bologna,

Via Irnerio 46, 40126 Bologna, ITALY

**Abstract**

In this paper we are going to consider a completely antisymmetric torsion to build metric-compatible connections in conformal gravity, in which then gauge potentials are added; this establishes a geometric background that will be filled with Dirac spinors: such kinematic construction shall be endowed with a system of field equations. The resulting dynamics will be worked out to yield the system of field equations in which torsional effects are converted into spinorial self-interactions: in particular the massless spinors shall display self-interactions of a specific form that enables them to have most of the features they have in the non-conformal theory but with the additional character of renormalizability. Particular behaviours and interactions will be investigated. Further remarks shall be sketched.

## Introduction

When developing a geometry based on any symmetry, both for its basic quantities and all of their derivatives, one is compelled to introduce an object called connection and indicated with $\Gamma^{A}_{B\mu}$ where the $A$ and $B$ indices refer to the underlying symmetry while the index $\mu$ refers to the spacetime; this is true in general, but for the spacetime itself the underlying symmetry is a spacetime symmetry, and since the connection $\Gamma^{\alpha}_{\beta\mu}$ has all indices of the same type it is possible to calculate its antisymmetric part $\Gamma^{\alpha}_{\beta\mu}-\Gamma^{\alpha}_{\mu\beta}=Q^{\alpha}_{\beta\mu}$ which in the most general instance is different from zero and it is called Cartan torsion tensor.

In the past, despite that in the most general case torsion is different from zero, there have been attempts to show that torsion should vanish because of some principle, such as the principle of equivalence [1]: the principle of equivalence spells the equivalence at a local level between frames and gravity in the sense that locally the effects of inertial accelerations can be simulated by gravitational acceleration, or equivalently that locally the effects of an inertial acceleration can compensate the gravitational acceleration; since the widely known Weyl theorem states that locally there exists a system of reference in which the

symmetric part of a connection can be vanished, the principle of equivalence can be realized by Weyl theorem, so long as we think the symmetric part of the connection as where the gravitational information is stored, if the connection has a single symmetric part, which is certainly the case if torsion vanishes, but we do not need to require so much as the vanishing of the whole torsion, since a completely antisymmetric torsion would be enough [2, 3, 4]. Therefore in order to have the principle of equivalence implemented we do not need to have the vanishing of torsion but only on the complete antisymmetry of torsion itself.

Another reason for which the spacetime is important is that, because of its dimensionality, the usual actions written in terms of curvature-squared terms are conformally invariant: conformal invariance is intriguing because gravitational or gauge actions are uniquely defined by the conformal invariance, as discussed by Weyl, the scale symmetry is related to the property of renormalizability, as discussed in [5], their projective structure allows to address the cosmological constant problem without dark energy, as in [6, 7], and to fit the galactic rotation curves without any form of dark matter, as in [8, 9]; a theory of gravitation with conformal invariance might suffer the problem that the universe appears not to possess such a symmetry, but these arguments can be circumvented as soon as a mechanism of gravitational spontaneous conformal symmetry breaking is introduced, as it has been addressed in [10] and more in detail in [11].

If we want to stay in the most general situation, then torsion has to be considered, and if we want to implement the principle of equivalence a completely antisymmetric torsion is to be taken; and if we want to study the peculiar circumstances provided by the conformal symmetry, then the conformal invariance must be implemented as well: so completely antisymmetric torsion with conformal transformation must be assigned. There are two types of conformal transformations, as discussed in [12], that is the *strong* conformal transformation, which is the case in which torsion is assumed to have a general conformal transformation of the form discussed in [13], and the *weak* conformal transformation, which is the case in which torsion does not have a conformal transformation compatibly with the fact that torsion is essentially independent from the metric for which the conformal properties are defined: in fact the strong conformal transformation is entirely loaded onto the trace vector part, and since in a conformal theory it is meaningless to require the vanishing of something that is not conformally invariant then it is impossible to have a vanishing trace torsion vector, and therefore this case would no be compatible with the irreducible completely antisymmetric torsion tensor we want to consider; completely antisymmetric torsions can only be compatible with the *weak* conformal transformations, that is no conformal transformation at all. In the present paper, we shall study the properties of a completely antisymmetric torsion tensor conformally invariant.

# 1 Axial Torsion in Conformal Gravity with Electrodynamics for Dirac Spinors

In the present construction we refer for the introduction of the general formalism and definitions to [13], although in the following we will anyway recall many of the most important concepts. We begin by recalling to the reader that the completely antisymmetric torsion is also called axial torsion because, once the

completely antisymmetric Levi-Civita tensor $\varepsilon_{\alpha\nu\pi\sigma}$ is introduced, the completely antisymmetric torsion $Q_{\rho\beta\mu}$ can equivalently be written in terms of an axial torsion vector $W^{\theta}$ according to the relationship given by

$$Q_{\alpha\mu\nu} = \varepsilon_{\alpha\mu\nu\sigma} W^{\sigma} \tag{1}$$

so that, taking into account the usual Levi-Civita connection $\Lambda^{\alpha}_{\beta\mu}$ we have

$$\Gamma^{\alpha}_{\beta\mu} = \frac{1}{2} g^{\alpha\rho} \varepsilon_{\rho\beta\mu\sigma} W^{\sigma} + \Lambda^{\alpha}_{\beta\mu} \tag{2}$$

as the most general decomposition of the connection we will consider in this paper, and as we have anticipated, the torsion $Q^{\alpha}{}_{\beta\mu}$ originally defined with the first upper index and the last two lower indices has no conformal transformation while the metric has the usual form of conformal transformation, so that the most general connection and the simplest Levi-Civita connection have the same conformal transformation; the most general connection defines the most general covariant derivative $D_{\mu}$ while the Levi-Civita connection defines the Levi-Civita covariant derivative $\nabla_{\mu}$ for which we have that the following relationships $D_{\mu} g_{\alpha\rho} = \nabla_{\mu} g_{\alpha\rho} = 0$ and $D_{\mu} \varepsilon_{\rho\beta\mu\sigma} = \nabla_{\mu} \varepsilon_{\rho\beta\mu\sigma} = 0$ hold and they are called metric-compatibility conditions: because of the connection decomposed according to (2) and these conditions, we have that the connection has its only symmetric part entirely written in terms of the metric, and therefore the local system of coordinates in which the symmetric connection vanishes and that where the metric is constant are the same, so that we may interpret the metric as what contains the gravitational information, and Weyl theorem expresses the principle of equivalence as we desired, and as was discussed in [2, 3, 4].

Now an equivalent formalism may be introduced, in which the metric is written as $g_{\alpha\nu} = e^{p}_{\alpha} e^{i}_{\nu} \eta_{pi}$ in terms of the constant Minkowskian metric $\eta_{ij}$ and a basis of vierbein $e^{i}_{\alpha}$ and such that with them, the previous connection (2) can be transformed into the spin-connection $\omega^{ip}{}_{\alpha}$ according to

$$\omega^{i}{}_{p\alpha} = e^{i}_{\sigma} (\Gamma^{\sigma}_{\rho\alpha} e^{\rho}_{p} + \partial_{\alpha} e^{\sigma}_{p}) \tag{3}$$

where the relationship $\omega^{ip}{}_{\alpha} = -\omega^{pi}{}_{\alpha}$ states the antisymmetry of the spin-connection; the relationship (3) and the antisymmetry of the spin-connection are respectively related to $D_{\mu} e^{k}_{\alpha} = \nabla_{\mu} e^{k}_{\alpha} = 0$ and $D_{\mu} \eta_{ij} = \nabla_{\mu} \eta_{ij} = 0$ known as coordinate-Lorentz compatibility conditions, since they establish that the previously defined coordinate formalism with Greek indices and the present formalism with Latin indices are such that at any differential level they are equivalent.

The reason for which such an equivalent but different formalism has been introduced is that with it the most general coordinate transformations are replaced by the Lorentz transformations, and since they have a specific form then they can be written explicitly, and possibly in terms of other representations such as the complex one, and since complex representation of the Lorentz transformation will act on complex fields, then the geometry of complex fields is given in terms of a connection known as gauge connection $A_{\mu}$ that will serve to introduce the gauge covariant derivatives; we are not going to spend more words on this subject as this is what is usually done in abelian gauge theories.

Now, an explicit representation of the complex Lorentz transformation can be achieved through the introduction of the $\boldsymbol{\gamma}_a$ matrices verifying the Clifford algebra $\{\boldsymbol{\gamma}_a, \boldsymbol{\gamma}_b\} = 2\mathbb{I}\eta_{ab}$ from which it is possible to define the $\boldsymbol{\sigma}_{ab}$ matrices

defined to be $\boldsymbol{\sigma}_{ab}=\frac{1}{4}[\boldsymbol{\gamma}_a,\boldsymbol{\gamma}_b]$ such that $\{\boldsymbol{\gamma}_a,\boldsymbol{\sigma}_{bc}\}=i\varepsilon_{abcd}\boldsymbol{\gamma}\boldsymbol{\gamma}^d$ as the complex generators of the complex representation of the Lorentz algebra, and from which with (3) it is possible to define the most general spinorial-connection as

$$\boldsymbol{\Omega}_\rho=\frac{1}{2}\omega^{ij}{}_\rho\boldsymbol{\sigma}_{ij}-iqA_\rho\mathbb{I} \tag{4}$$

in terms of the parameter $q$ known as the charge of the spin-$\frac{1}{2}$ spinorial field, and through this spinorial connection we may define the spinorial covariant derivatives $\boldsymbol{D}_\mu$ as usual; then $\boldsymbol{D}_\mu\boldsymbol{\gamma}_a=\boldsymbol{\nabla}_\mu\boldsymbol{\gamma}_a=0$ is automatically accomplished.

From the connection (2) it is also possible to define the curvature as

$$G^\rho{}_{\xi\mu\nu}=\partial_\mu\Gamma^\rho_{\xi\nu}-\partial_\nu\Gamma^\rho_{\xi\mu}+\Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\xi\nu}-\Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\xi\mu} \tag{5}$$

antisymmetric in both the first and second pair of indices, and thus with one contraction given as $G^\rho{}_{\mu\rho\nu}=G_{\mu\nu}$ with contraction $G_{\eta\nu}g^{\eta\nu}=G$ and it is also possible to define the purely metric curvature as given by the following

$$R^\rho{}_{\xi\mu\nu}=\partial_\mu\Lambda^\rho_{\xi\nu}-\partial_\nu\Lambda^\rho_{\xi\mu}+\Lambda^\rho_{\sigma\mu}\Lambda^\sigma_{\xi\nu}-\Lambda^\rho_{\sigma\nu}\Lambda^\sigma_{\xi\mu} \tag{6}$$

antisymmetric in both the first and second pair of indices and symmetric for a switch of the first with the second pair of indices, with one contraction given according to $R^\rho{}_{\mu\rho\nu}=R_{\mu\nu}$ symmetric with contraction $R_{\eta\nu}g^{\eta\nu}=R$ such that

$$\begin{aligned}G_{\rho\xi\mu\nu}=R_{\rho\xi\mu\nu}+\tfrac{1}{2}\left(\varepsilon_{\rho\xi\nu\alpha}\nabla_\mu W^\alpha-\varepsilon_{\rho\xi\mu\alpha}\nabla_\nu W^\alpha\right)-\\-\tfrac{1}{4}(W_\rho W_{[\mu}g_{\nu]\xi}-W_\xi W_{[\mu}g_{\nu]\rho})+\tfrac{1}{8}W^2(g_{\rho[\mu}g_{\nu]\xi}-g_{\xi[\mu}g_{\nu]\rho})\end{aligned} \tag{7}$$

is the decomposition in terms of the axial torsion and where the connection conformal transformations induces the curvature conformal transformations themselves, and it is called Riemann curvature tensor: now from these curvatures it is possible to define other curvatures that have the same symmetries for indices transposition but which are irreducible according to

$$W_{\alpha\beta\mu\nu}=G_{\alpha\beta\mu\nu}-\tfrac{1}{2}(G_{\alpha[\mu}g_{\nu]\beta}-G_{\beta[\mu}g_{\nu]\alpha})+\tfrac{1}{12}G(g_{\alpha[\mu}g_{\nu]\beta}-g_{\beta[\mu}g_{\nu]\alpha}) \tag{8}$$

and also according to the purely metric counterpart

$$C_{\alpha\beta\mu\nu}=R_{\alpha\beta\mu\nu}-\tfrac{1}{2}(R_{\alpha[\mu}g_{\nu]\beta}-R_{\beta[\mu}g_{\nu]\alpha})+\tfrac{1}{12}R(g_{\alpha[\mu}g_{\nu]\beta}-g_{\beta[\mu}g_{\nu]\alpha}) \tag{9}$$

related through the decomposition

$$W_{\alpha\beta\mu\nu}=C_{\alpha\beta\mu\nu}+\tfrac{1}{2}\left[\varepsilon_{\alpha\beta\theta[\mu}g_{\nu]\rho}-\tfrac{1}{2}\left(\varepsilon_{\alpha\rho\theta[\mu}g_{\nu]\beta}-\varepsilon_{\beta\rho\theta[\mu}g_{\nu]\alpha}\right)\right]\nabla^\rho W^\theta \tag{10}$$

in which due to the derivatives of the axial torsion, $W_{\alpha\beta\mu\nu}$ is not conformally invariant even though $C_{\alpha\beta\mu\nu}$ is conformally invariant, and this is what is known as the Weyl curvature conformal tensor. The reason for this has to be tracked back to the implicit presence of torsion within the curvature, and in particular to the fact that there are derivatives of torsion, beside squared torsion terms, in the curvature, and thus in the irreducible curvature itself: whatever conformal transformation is used, the derivatives of torsion would always transform, spoiling for $W_{\alpha\beta\mu\nu}$ the conformal invariance that for $C_{\alpha\beta\mu\nu}$ was ensured.

Now a solution could be obtained by adding to the curvature $G_{\alpha\beta\mu\nu}$ some piece in terms of torsion getting a modified torsional-curvature $M_{\alpha\beta\mu\nu}$ in which

the simultaneous presence of torsion implicitly through $G_{\alpha\beta\mu\nu}$ and explicitly added would provide the cancellation of all extra terms during the conformal transformation, leaving $M_{\alpha\beta\mu\nu}$ such that its irreducible part $T_{\alpha\beta\mu\nu}$ would be conformally invariant: in [13] the solution we found for strong conformal transformations consisted in adding to $G_{\alpha\beta\mu\nu}$ some *squared* torsional contributions, in order to get an $M_{\alpha\beta\mu\nu}$ in which the non-conformal transformation of the derivatives of torsion and that of all squared torsion contributions combined cancelled away, so to have $T_{\alpha\beta\mu\nu}$ conformally invariant; but in the present paper there is no trace torsion vector nor conformal transformation that would provide that particular squared torsion correction nor any possible cancellation, and therefore we can only add to $G_{\alpha\beta\mu\nu}$ some *derivatives* of torsion, beside eventual *squared* torsion pieces in terms of $a$ and $b$ parameters, as in the form

$$\begin{aligned} M_{\alpha\beta\mu\nu} = G_{\alpha\beta\mu\nu} + \tfrac{1}{2}(\varepsilon_{\alpha\beta\mu\theta}D_{\nu}W^{\theta} - \varepsilon_{\alpha\beta\nu\theta}D_{\mu}W^{\theta}) + \\ + \tfrac{a}{4}(W_{\alpha}W_{[\mu}g_{\nu]\beta} - W_{\beta}W_{[\mu}g_{\nu]\alpha}) + \tfrac{b}{8}W^{2}(g_{\alpha[\mu}g_{\nu]\beta} - g_{\beta[\mu}g_{\nu]\alpha}) \end{aligned} \tag{11}$$

whose irreducible part becomes

$$T_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu}$$

where the derivatives of torsion cancel out, independently on all squared torsion contributions in terms of $a$ and $b$ parameters, and so $T_{\alpha\beta\mu\nu}$ is indeed conformally invariant, although the fact that it reduces to $C_{\alpha\beta\mu\nu}$ that is the Weyl curvature conformal tensor implies that the correction is somewhat trivial. Therefore it seems that in the case of axial torsion with no conformal transformations, the most general curvature that is conformally invariant is the Weyl curvature itself.

Before proceeding, we have also to recall that for the gauge connection the gauge curvature is defined as $F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu}$ antisymmetric and thus irreducible, and as the gauge connection is usually assumed to have a trivial conformal transformation it follows that the gauge curvature is conformally invariant automatically; notice also that this definition of gauge curvature does not have to be generalized up to the definition given in terms of the full connection, according to the discussion outlined in [4], and so there is no torsion in the gauge curvature, and torsion and gauge curvature are independent as well.

Finally, what transforms according to the complex representation of the Lorentz transformation, or spinorial transformation, is defined to be the spinor field $\psi$ whose conformal transformation is given by $\psi \rightarrow \sigma^{-\frac{3}{2}}\psi$ as usual.

## 2 Cartan-Weyl-Maxwell-Dirac Field Equations

We have seen in the previous section that all effort to follow the line of [13] and get a curvature that is conformally invariant, in the case of axial torsion with no conformal transformations brings the most general of such curvatures back to the torsionless curvature with conformal invariance known as Weyl tensor, and so Cartan torsion and Weyl curvature are independently conformally invariant and separated at a kinematic level. However, if in [13] the strong conformal transformations allowed us to employ torsion to modify the Weyl curvature as to achieve its conformal invariance but also forbade torsion to explicitly appear in the action without spoiling its conformal invariance, here the weak conformal transformation that keeps Cartan torsion and Weyl curvature separated also

allows Cartan torsion to explicitly enter beside Weyl curvature in an action that is still conformally invariant; nonetheless, it is easy to see that there is no possible product of Cartan torsions and Weyl curvatures that could enter in the action preserving its conformal invariance, and Cartan torsion and Weyl curvature will be separated also dynamically. To define the dynamical features of the model, we have to take into account the fact that no derivative of the axial torsion should characterize the action: the first reason for this is that we have already seen that derivatives of the axial torsion are not in general conformally invariant, although this problem might be circumvented by the fact that there are many of such term, each entering with its own coupling factor, for which a fine-tuning may restore conformal invariance; a second reason for this consists in the fact that any derivative of torsion would result into dynamical torsional field equations, allowing torsion to propagate out of matter, but there are stringent limits for the presence of torsion in vacuum [14]. If we want no derivative of torsion in the action, then $|W^{\alpha}W_{\alpha}|^{2}$ is the only axial torsion term that defines an action with conformal invariance; the term $C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$ defines the action with conformal invariance for gravitation as usual: therefore we have that

$$S_{\mathrm{gravity}}=\int\tfrac{1}{4}\left(3k_{Q}W^{\alpha}W_{\alpha}W^{\rho}W_{\rho}+k_{G}C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}\right)\sqrt{|g|}dV \tag{12}$$

in terms of the two constants $k_{Q}$ and $k_{G}$ is a gravitational action that is also conformally invariant, and it is the most general under the hypotheses above.

For the gauge field we have that the term $F_{\mu\nu}F^{\mu\nu}$ is the only one that can be added into the action, which will therefore be given by

$$S_{\mathrm{electrodynamic}}=-\int\left(\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{|g|}dV \tag{13}$$

and this is the electrodynamic action and it is conformally invariant trivially.

Finally, the Dirac spinorial action is given as usual by

$$S_{\mathrm{matter}}=\int[\tfrac{i\hbar}{2}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{D}_{\rho}\psi-\boldsymbol{D}_{\rho}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi)]|e|dV \tag{14}$$

and this is the material action that in the massless case is conformally invariant.

Notice that this total action with conformal invariance is uniquely defined.

By varying the action with respect to the axial torsion one gets

$$k_{Q}W^{2}W_{\rho}=\tfrac{\hbar}{4}\overline{\psi}\boldsymbol{\gamma}_{\rho}\boldsymbol{\gamma}\psi \tag{15}$$

coupling the axial torsion to the axial spin density of the Dirac spinor in terms of the coupling constant $k_{Q}$, as the torsion-spin density field equations.

By varying with respect to the metric one gets

$$\begin{aligned}&-k_{Q}W^{2}\left(\tfrac{1}{4}g^{\mu\alpha}W^{2}-W^{\mu}W^{\alpha}\right)+\\&+k_{G}\left(\tfrac{1}{4}g^{\mu\alpha}C^{\theta\sigma\rho\beta}C_{\theta\sigma\rho\beta}-C^{\theta\sigma\rho\mu}C_{\theta\sigma\rho}{}^{\alpha}-C^{\mu\beta\alpha\nu}R_{\beta\nu}-2\nabla_{\beta}\nabla_{\nu}C^{\mu\beta\alpha\nu}\right)-\\&-\left(\tfrac{1}{4}g^{\mu\alpha}F^{2}-F^{\mu\rho}F^{\alpha}{}_{\rho}\right)=\tfrac{i\hbar}{4}\left(\overline{\psi}\boldsymbol{\gamma}^{\{\mu}\boldsymbol{\nabla}^{\alpha\}}\psi-\boldsymbol{\nabla}^{\{\alpha}\overline{\psi}\boldsymbol{\gamma}^{\mu\}}\psi\right)+\\&+\tfrac{\hbar}{4}\left(\tfrac{1}{2}\overline{\psi}\boldsymbol{\gamma}^{\{\alpha}\boldsymbol{\gamma}\psi W^{\mu\}}-\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\gamma}\psi W_{\rho}g^{\alpha\mu}\right)\end{aligned} \tag{16}$$

coupling the irreducible curvature to the traceless energy density of the spinor field in terms of the constant $k_{G}$, as the curvature-energy density field equations.

The variation with respect to the gauge potential gives

$$\nabla_{\mu}F^{\mu\nu}=q\hbar\overline{\psi}\boldsymbol{\gamma}^{\nu}\psi \tag{17}$$

coupling the derivative of the gauge strength to the current of the spinor field in terms of the charge $q$, as the gauge-current field equations.

And the variation with respect to the spinor field one obtains

$$i\hbar\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi+\tfrac{3\hbar}{4}W_{\sigma}\boldsymbol{\gamma}\boldsymbol{\gamma}^{\sigma}\psi=0 \tag{18}$$

as the massless spinorial field equations determining the behaviour of the massless spinor itself in terms of the constant $\hbar$, as the spinorial field equations.

As a final comment, we have to notice that in all field equations, the number of independent fields and degrees of freedom is matched consistently, and in particular we remark that the spinorial field equations have characteristic equation given by the simplest $n^2=0$ showing that the principle of causality is implemented correctly, according to the discussion reported in [15].

The whole system of field equations is called Cartan-Weyl-Maxwell-Dirac Field Equations, which we are next going to study.

## 2.1 The Bosonic Interactions

To begin our study of the previous system of field equations, we first notice that field equations (19) are a thoroughly new set of field equations, whose algebraic form implies that they are not field equations but constraints, so that consequently the torsion does not propagate out of matter as desired, and they can straightforwardly be inverted, so that they can be written in the form

$$W_{\rho}=\left(\tfrac{\hbar}{4k_{Q}}\right)^{\frac{1}{3}}\left(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\gamma}\psi\overline{\psi}\boldsymbol{\gamma}_{\mu}\boldsymbol{\gamma}\psi\right)^{-\frac{1}{3}}\overline{\psi}\boldsymbol{\gamma}_{\rho}\boldsymbol{\gamma}\psi \tag{19}$$

showing that the torsion tensor vanishes whenever the spinor fields themselves tend to approach zero: by employing this form of the torsion-spin coupling field equations, it is consequently be possible to have torsion substituted in terms of the spin density of the spinor field, so that all torsional contributions can be seen as spinorial field self-interactions, in all the remaining field equations.

Then field equations (20) become the usual field equations with extra terms

$$\begin{aligned}k_{G}\left(\tfrac{1}{4}g^{\mu\alpha}C^{\theta\sigma\rho\beta}C_{\theta\sigma\rho\beta}-C^{\theta\sigma\rho\mu}C_{\theta\sigma\rho}{}^{\alpha}-C^{\mu\beta\alpha\nu}R_{\beta\nu}-2\nabla_{\beta}\nabla_{\nu}C^{\mu\beta\alpha\nu}\right)-\\-\left(\tfrac{1}{4}g^{\mu\alpha}F^{2}-F^{\mu\rho}F^{\alpha}{}_{\rho}\right)=\tfrac{i\hbar}{4}\left(\overline{\psi}\boldsymbol{\gamma}^{\{\mu}\boldsymbol{\nabla}^{\alpha\}}\psi-\boldsymbol{\nabla}^{\{\alpha}\overline{\psi}\boldsymbol{\gamma}^{\mu\}}\psi\right)+\\-\tfrac{1}{4}g^{\alpha\mu}\left(\tfrac{27\hbar^{4}}{256k_{Q}}\right)^{\frac{1}{3}}\left(\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi\overline{\psi}\boldsymbol{\gamma}_{\mu}\psi\right)^{\frac{2}{3}}\end{aligned} \tag{20}$$

where the extra term expresses the spinorial field self-interactions vanishing when the spinors vanish; in this form of the curvature-energy coupling field equations the extra term has the form of the cosmological constant term.

Field equations (21) are the usual set of field equations

$$\nabla_{\mu}F^{\mu\nu}=q\hbar\overline{\psi}\boldsymbol{\gamma}^{\nu}\psi \tag{21}$$

as the gauge field equations we would have had in the standard case.

But the most interesting case is given by the field equations (22) becoming

$$i\hbar\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-\left(\tfrac{27\hbar^{4}}{256k_{Q}}\right)^{\frac{1}{3}}\left(\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi\overline{\psi}\boldsymbol{\gamma}_{\rho}\psi\right)^{-\frac{1}{3}}\overline{\psi}\boldsymbol{\gamma}_{\nu}\psi\boldsymbol{\gamma}^{\nu}\psi=0 \tag{22}$$

with massless spinorial self-interactions, and after the massless spinorial fields are split into left-handed and right-handed projections these massless spinorial self-interactions become left-handed and right-handed semi-spinorial mutual-interactions, consequently showing that the two chiral decompositions are intertwined dynamically precisely as if they were massive interactions.

Notice that although general conformal models in torsionless limit are restricted with respect to the purely metric one [16], the present model in torsionless limit and the purely metric one are actually the same.

## 2.2 The Fermionic Behaviour

Consider that field equations (22) can be written as

$$i\hbar\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-\left(\tfrac{27\hbar^{4}}{256k_{Q}}\right)^{\frac{1}{3}}\left(\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi\overline{\psi}\boldsymbol{\gamma}_{\rho}\psi\right)^{-\frac{1}{3}}\overline{\psi}\boldsymbol{\gamma}_{\nu}\psi\boldsymbol{\gamma}^{\nu}\psi=0 \tag{23}$$

but after a simple Fierz rearrangement also as

$$i\hbar\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-\left(\tfrac{27\hbar^{4}}{256k_{Q}}\right)^{\frac{1}{3}}\left[\left(\overline{\psi}\psi\right)^{2}+\left(i\overline{\psi}\boldsymbol{\gamma}\psi\right)^{2}\right]^{-\frac{1}{3}}\left[\left(\overline{\psi}\psi\right)\mathbb{I}+i\left(i\overline{\psi}\boldsymbol{\gamma}\psi\right)\boldsymbol{\gamma}\right]\psi=0 \tag{24}$$

with massless spinorial self-interactions that are not only conformally invariant but also chirally invariant; from the conformal invariance we immediately have their renormalizability while from chiral invariance we know we can always choose the chiral gauge to give $i\overline{\psi}\boldsymbol{\gamma}\psi=0$ so that (24) reduces to

$$i\hbar\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-\left(\tfrac{27\hbar^{4}}{256k_{Q}}\right)^{\frac{1}{3}}\left(\overline{\psi}\psi\right)^{\frac{1}{3}}\psi=0 \tag{25}$$

showing that the extra term has the form of a potential of chiral gauge spontaneous symmetry breaking, as those discussed in reference [17]: in particular, we also have to notice that these interactions are able to entail a dynamical form of the principle of exclusion, as discussed in [18]. Also notice that in the weak field approximation where $\overline{\psi}\psi\rightarrow 0$ the interaction disappears, showing that the weak-field case is recovered consistently, although no slow-speed regime can actually be obtained because of the intrinsic masslessness of the spinor field.

Two of the most important consequences concern the behaviour of the solutions under certain rotationally invariant configurations and that of two spinors interacting between each other, as we are going to discuss.

### 2.2.1 Rotationally-Symmetric Backgrounds

The system of Dirac field equations is expected to display solutions that are localized, and accordingly stable compact solutions should find place in stationary spherically symmetric spacetimes: in the past there have been discussions on this physical situation, and in such backgrounds Dirac matter fields have been considered in [19, 20, 21, 22], with gauge potentials in [23, 24], with the extension to gravity in [25] and with the inclusion of torsion in [26] as torsion gives precisely rise to the non-linear self-interactions of the Dirac matter field.

In the present section, we are going to consider a background with stationary spherical symmetry, employing the Lie derivatives to see what is the most general restrictions that are imposed on the field equations and their fields in

this configuration: first of all, it is clear that in stationary spherically symmetric backgrounds there can be no angular component for any vector appearing in a field equation, in particular implying that the axial torsion only has the temporal and radial components $W_t$ and $W_r$ that are present; then, in the frame at rest with respect to the origin of coordinates, the metric is given by

$$g_{tt} = A^2 \qquad g_{rr} = -B^2 \qquad g_{\theta\theta} = -r^2 \qquad g_{\varphi\varphi} = -r^2(\sin\theta)^2 \tag{26}$$

while the tetrads are given by

$$e^0_t = A \qquad e^1_r = B \qquad e^2_\theta = r \qquad e^3_\varphi = r\sin\theta \tag{27}$$

in terms of two functions of the radial coordinate $A$ and $B$ and where the gamma matrices are in chiral representation; the electrodynamic potential too will have at most only the temporal and radial components, although through a deeper analysis it is possible to show that only the component

$$A_t = \phi \tag{28}$$

in term of the function of the radial coordinate $\phi$ can give electrostatic radial configurations; finally the spinor field will be taken in the general form

$$\psi = \left(\begin{array}{c} \rho e^{i\frac{\alpha}{2}} \\ \rho e^{i\frac{\alpha}{2}} \\ \eta e^{i\frac{\beta}{2}} \\ \eta e^{i\frac{\beta}{2}} \end{array}\right) \tag{29}$$

because this is the most general form for which the corresponding bilinear fields

$$\overline{\psi}\boldsymbol{\gamma}^0\psi = \overline{\psi}\boldsymbol{\gamma}^1\boldsymbol{\gamma}\psi = 2(\eta^2 + \rho^2) \tag{30}$$
$$\overline{\psi}\boldsymbol{\gamma}^1\psi = \overline{\psi}\boldsymbol{\gamma}^0\boldsymbol{\gamma}\psi = 2(\eta^2 - \rho^2) \tag{31}$$
$$i\overline{\psi}\boldsymbol{\gamma}\psi = 4\eta\rho\sin\left(\tfrac{\alpha-\beta}{2}\right) \tag{32}$$
$$\overline{\psi}\psi = 4\eta\rho\cos\left(\tfrac{\alpha-\beta}{2}\right) \tag{33}$$

where the product $\eta\rho$ and the difference $\alpha - \beta$ depend on the radial coordinate alone are the most general to be compatible with the rotational invariance.

Due to the conformal invariance of the metric background, it is known that we may always choose a conformal gauge in which $AB \equiv 1$ which, although not really necessary, it would simplify the notations a bit; nevertheless having exploited the degrees of freedom of the conformal scaling, there will be no way to employ the conformal transformations for any further simplification.

Finally, we notice that it is also possible to employ the gauge and chiral transformations to set $\alpha \equiv \beta \equiv 0$ providing powerful simplifications.

We see that from the spinorial bilinears above and with these symmetries, the additional scalar and pseudo-scalar bilinears are given to verify the relationships

$$\overline{\psi}\boldsymbol{\gamma}^a\psi\overline{\psi}\boldsymbol{\gamma}_a\psi = -\overline{\psi}\boldsymbol{\gamma}^a\boldsymbol{\gamma}\psi\overline{\psi}\boldsymbol{\gamma}_a\boldsymbol{\gamma}\psi = \left(\overline{\psi}\psi\right)^2 = 16\eta^2\rho^2 \tag{34}$$

showing again that everything depends only on the radial coordinate alone, and from which it is clear that in order to preserve the torsionally induced spinorial self-interactions in the field equations the conditions $\overline{\psi}\psi \neq 0$ will have to be

imposed, which would become the conditions $\eta \neq 0$ and $\rho \neq 0$ to be taken as the conditions upon which the entire reasoning we are going to follow will rely.

By plugging these fields into the fermionic field equations we get them split into the left-hand and right-hand projections given by the expressions

$$\begin{aligned}&\left(\tfrac{27\hbar^4\eta\rho}{64k_Q}\right)^{\frac{1}{3}}\cos\left(\tfrac{\alpha-\beta}{2}\right)\left(\eta e^{i\frac{\beta}{2}}\right)-i\left(\tfrac{27\hbar^4\eta\rho}{64k_Q}\right)^{\frac{1}{3}}\sin\left(\tfrac{\alpha-\beta}{2}\right)\left(-\eta e^{i\frac{\beta}{2}}\right)-\\&-\tfrac{1}{\sqrt{Ar^2}}\left(\tfrac{i}{A}\left(iq\phi+\tfrac{\partial}{\partial t}\right)\left(-\sqrt{Ar^2}\rho e^{i\frac{\alpha}{2}}\right)+\tfrac{i}{B}\tfrac{\partial}{\partial r}\left(\sqrt{Ar^2}\rho e^{i\frac{\alpha}{2}}\right)\right)-\\&-\tfrac{1}{r}\left[\tfrac{1}{\sqrt{\sin\theta}}\left(\tfrac{\partial}{\partial\theta}\left(\sqrt{\sin\theta}\rho e^{i\frac{\alpha}{2}}\right)+\tfrac{i}{\sin\theta}\tfrac{\partial}{\partial\varphi}\left(\sqrt{\sin\theta}\rho e^{i\frac{\alpha}{2}}\right)\right)\right]=0\end{aligned}\qquad(35)$$

$$\begin{aligned}&\left(\tfrac{27\hbar^4\eta\rho}{64k_Q}\right)^{\frac{1}{3}}\cos\left(\tfrac{\alpha-\beta}{2}\right)\left(-\eta e^{i\frac{\beta}{2}}\right)-i\left(\tfrac{27\hbar^4\eta\rho}{64k_Q}\right)^{\frac{1}{3}}\sin\left(\tfrac{\alpha-\beta}{2}\right)\left(\eta e^{i\frac{\beta}{2}}\right)-\\&-\tfrac{1}{\sqrt{Ar^2}}\left(\tfrac{i}{A}\left(iq\phi+\tfrac{\partial}{\partial t}\right)\left(\sqrt{Ar^2}\rho e^{i\frac{\alpha}{2}}\right)+\tfrac{i}{B}\tfrac{\partial}{\partial r}\left(-\sqrt{Ar^2}\rho e^{i\frac{\alpha}{2}}\right)\right)-\\&-\tfrac{1}{r}\left[\tfrac{1}{\sqrt{\sin\theta}}\left(\tfrac{\partial}{\partial\theta}\left(\sqrt{\sin\theta}\rho e^{i\frac{\alpha}{2}}\right)+\tfrac{i}{\sin\theta}\tfrac{\partial}{\partial\varphi}\left(\sqrt{\sin\theta}\rho e^{i\frac{\alpha}{2}}\right)\right)\right]=0\end{aligned}\qquad(36)$$

$$\begin{aligned}&\left(\tfrac{27\hbar^4\eta\rho}{64k_Q}\right)^{\frac{1}{3}}\cos\left(\tfrac{\alpha-\beta}{2}\right)\left(-\rho e^{i\frac{\alpha}{2}}\right)-i\left(\tfrac{27\hbar^4\eta\rho}{64k_Q}\right)^{\frac{1}{3}}\sin\left(\tfrac{\alpha-\beta}{2}\right)\left(-\rho e^{i\frac{\alpha}{2}}\right)-\\&-\tfrac{1}{\sqrt{Ar^2}}\left(\tfrac{i}{A}\left(iq\phi+\tfrac{\partial}{\partial t}\right)\left(\sqrt{Ar^2}\eta e^{i\frac{\beta}{2}}\right)+\tfrac{i}{B}\tfrac{\partial}{\partial r}\left(\sqrt{Ar^2}\eta e^{i\frac{\beta}{2}}\right)\right)-\\&-\tfrac{1}{r}\left[\tfrac{1}{\sqrt{\sin\theta}}\left(\tfrac{\partial}{\partial\theta}\left(\sqrt{\sin\theta}\eta e^{i\frac{\beta}{2}}\right)+\tfrac{i}{\sin\theta}\tfrac{\partial}{\partial\varphi}\left(\sqrt{\sin\theta}\eta e^{i\frac{\beta}{2}}\right)\right)\right]=0\end{aligned}\qquad(37)$$

$$\begin{aligned}&\left(\tfrac{27\hbar^4\eta\rho}{64k_Q}\right)^{\frac{1}{3}}\cos\left(\tfrac{\alpha-\beta}{2}\right)\left(\rho e^{i\frac{\alpha}{2}}\right)-i\left(\tfrac{27\hbar^4\eta\rho}{64k_Q}\right)^{\frac{1}{3}}\sin\left(\tfrac{\alpha-\beta}{2}\right)\left(\rho e^{i\frac{\alpha}{2}}\right)-\\&-\tfrac{1}{\sqrt{Ar^2}}\left(\tfrac{i}{A}\left(iq\phi+\tfrac{\partial}{\partial t}\right)\left(-\sqrt{Ar^2}\eta e^{i\frac{\beta}{2}}\right)+\tfrac{i}{B}\tfrac{\partial}{\partial r}\left(-\sqrt{Ar^2}\eta e^{i\frac{\beta}{2}}\right)\right)-\\&-\tfrac{1}{r}\left[\tfrac{1}{\sqrt{\sin\theta}}\left(\tfrac{\partial}{\partial\theta}\left(\sqrt{\sin\theta}\eta e^{i\frac{\beta}{2}}\right)+\tfrac{i}{\sin\theta}\tfrac{\partial}{\partial\varphi}\left(\sqrt{\sin\theta}\eta e^{i\frac{\beta}{2}}\right)\right)\right]=0\end{aligned}\qquad(38)$$

as it is easy to check with a direct substitution, and by adding (35) to (36) and equation (37) to (38) one gets in particular the angular field equations given by

$$\tfrac{\partial}{\partial\theta}\left(\sqrt{\sin\theta}\rho e^{i\frac{\alpha}{2}}\right)+\tfrac{i}{\sin\theta}\tfrac{\partial}{\partial\varphi}\left(\sqrt{\sin\theta}\rho e^{i\frac{\alpha}{2}}\right)=0\qquad(39)$$

$$\tfrac{\partial}{\partial\theta}\left(\sqrt{\sin\theta}\eta e^{i\frac{\beta}{2}}\right)+\tfrac{i}{\sin\theta}\tfrac{\partial}{\partial\varphi}\left(\sqrt{\sin\theta}\eta e^{i\frac{\beta}{2}}\right)=0\qquad(40)$$

in which the gauge $\alpha \equiv \beta \equiv 0$ would then provide the condition

$$\tfrac{\partial}{\partial\theta}\left(\sin\theta\rho\eta\right)=0\qquad(41)$$

which have to be valid with $\eta \neq 0$ and $\rho \neq 0$ simultaneously; this condition becomes $\cos\theta = 0$ since the term $\eta\rho$ does not depend on any angle: this is impossible for generic values of the elevation angle, and a contradiction follows.

It is possible to extend these results to the case in which static spherically symmetric spacetimes are not only isotropic with respect to a single point but to every point and therefore also homogeneous: in this case, we have that the bilinear fields $\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi$ and $\overline{\psi}\boldsymbol{\gamma}_{\mu}\boldsymbol{\gamma}\psi$ may only possess the temporal components in order to be compatible with the rotational and translational invariance.

Now the spinorial identity $\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi\overline{\psi}\boldsymbol{\gamma}_{\mu}\psi+\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\gamma}\psi\overline{\psi}\boldsymbol{\gamma}_{\mu}\boldsymbol{\gamma}\psi\equiv 0$ immediately gives us that $0\equiv\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi\overline{\psi}\boldsymbol{\gamma}_{\mu}\psi+\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\gamma}\psi\overline{\psi}\boldsymbol{\gamma}_{\mu}\boldsymbol{\gamma}\psi\equiv|\overline{\psi}\boldsymbol{\gamma}^{0}\psi|^2+|\overline{\psi}\boldsymbol{\gamma}^{0}\boldsymbol{\gamma}\psi|^2$ implying

in particular that $0 = \overline{\psi}\boldsymbol{\gamma}^0\psi \equiv \psi^\dagger\psi$ but because the quantity $\psi^\dagger\psi = 0$ is strictly positive defined we have that the solution $\psi = 0$ alone is ultimately possible.

It is even easier to see how we can extend these results to the case in which isotropic-homogeneous spacetimes are in particular Minkowskian spacetimes, as in this case no dynamical consideration nor algebraic argument is needed, since the very presence of the two vectors $\overline{\psi}\boldsymbol{\gamma}^\mu\psi$ and $\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}\psi$ is enough to see that the system is not compatible with the Poincaré symmetry of the spacetime.

This result can be summarized in a no-go theorem stating that rotationally and rototraslationally invariant and even flat backgrounds are all incompatible with spinorial self-interacting fields when all interactions are accounted.

This theorem has already been demonstrated for the non-conformal Dirac equation in reference [27] where we also remarked that it might not be valid for general conformal Dirac equations [28], although it happens to be recovered for the special conformal Dirac equation we have obtained here.

#### 2.2.2 Weakly-Interacting Matter

The model we have presented here gives rise to a Dirac equations with renormalizable chiral interactions structurally analogous to those present in the standard model, and the fact that these interactions in the standard model are due to the $SU(2)_L$ gauge symmetry while here they come from torsion makes it possible to think that if the $SU(2)_L$ gauge fields are not after all fundamental then they may be composite in terms of torsion: in the past this situation was actually conjectured, for instance as discussed in references [29, 30, 31, 32, 33, 34].

Quite recently, the fact that from torsionally-induced interactions it is indeed possible to get weak-like forces has been proven for the non-conformal Dirac equation in references [35, 36, 37] and due to the particular structure of the torsion interaction such weak force can be obtained for the special conformal Dirac equation we have obtained here as well.

## Conclusion

In this paper, we have considered axial torsion with no conformal transformations and we have insisted on the fact that no derivatives of the axial torsion should be present in the action; the metric was taken with the standard conformal transformation and implemented in the action in terms of the purely metric curvature conformal tensor; gauge conformal fields were added; massless spinors were taken eventually: with this field content, a unique action invariant under conformal transformations was obtained, in which the absence of any coupling between torsion and metric curvature or derivative of torsion provided torsional field equations in which there was no metric curvature and torsion appeared algebraically, so that torsion did not propagate out of matter and they could be inverted as to make it possible to have torsion substituted in terms of the spin density of the spinor field in all other field equations; the curvature field equations were given in terms of the energy density of the spinor field plus extra terms proportional to the spinor field bilinears and having the form of the cosmological constant term; the gauge field equations were unmodified as it might have been expected; the Dirac equations were given in terms of massless spinorial self-interactions for which the two chiral decompositions were inter-

twined dynamically as for massive interactions. We have seen that for the Dirac equations, the spinorial self-interactions gave rise to non-linearities that were incompatible with rotationally-symmetric backgrounds; and that these interactions could be written with a structure that was formally equivalent to that of the weak forces: these results were already known for the non-conformal model and in this paper they have been extended to the specific conformal model presented here. Further questions regarding the extension to the standard model of cosmology and particle physics are already being addressed.

In developing the most general geometric background that is supposed to constitute the basis upon which to construct physics and if the geometry is required to be conformal so to have a physics that is renormalizable, then one is compelled to consider the axial torsion with no conformal transformations, and if torsion is to have no propagation out of matter as to be compatible with all observations, then one is forced to have no derivatives of the axial conformal torsion in the action; if one wishes to consider renormalizable physics with the proper torsionless limit, then these are unavoidable requirements that alone would make this model intriguing: but moreover, these requirements lead toward a uniquely defined action that yields renormalizable field equations whose solutions are those we would have had in the torsionless case plus additional spinorial self-interactions, providing non-linearities that are capable of endowing the system with very interesting consequences. In view of these considerations, this model clearly appears to be one of the simplest and yet most powerful alternatives to the standard theory of gravity with electrodynamics for Dirac spinors, and it certainly deserves further attention.

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